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THE LIMIT PERIODIC MOTIONS OF CERTAIN SYSTEMS WITH AFTEREFFECT†

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Systems with aftereffect, the state of which is given by non-linear Volterra-type integro-differential equations with small perturbations, are investigated. The question of the existence of limit periodic motions in such systems is considered, assuming that the linearized unperturbed system is asymptotically stable and that the perturbations, and also the non-linear terms, contain functions of time, which tend exponentially to periodic functions. The limit periodic motions of a rigid plate (the model of a wing) when there is an unsteady air flow about it are considered as an example.

1. LIMIT PERIODIC MOTIONS

We will consider a system with aftereffect, described by the following Volterra integro-differential equation

$$\frac{dx}{dt} = Ax + \int_0^t K(t-s)x(s)ds + F(x, y, z, t) + \mu(\Phi(t) + D_1(t)x + D_2(t)y + D_3(t)z)$$
$$x, y, z \in R^n, \quad x = \mathrm{col}(x_1, \ldots, x_n) \tag{1.1}$$

in which A is a constant $n \times n$ matrix, $\Phi(t) = \Phi_p(t) + \Phi_e(t)$ is a vector function which is continuous when $t \in R^+$, where $\Phi_p(t)$ is its periodic part: $\Phi_p(t + T) = \Phi_p(t)$ and $\Phi_e(t) \to 0$ exponentially as $t \to +\infty$, $\mu \geq 0$ is a small parameter, $D_i(t) = D_{ip}(t) + D_{ie}(t)$ $(t = 1, 2, 3)$ are matrices with elements similar to $\Phi(t)$, and $F(x, y, z, t)$ is a function, continuous in $t \in R^+$, belonging to the class C^1 with respect to x, y, z from a certain neighbourhood

$$B(x, y, z) = \{x, y, z \in R^n: \|x\|, \|y\|, \|z\| < \delta_1\}$$

The continuous $n \times n$ matrix $K(t)$ is specified when $t > 0$ and satisfies the inequality

$$\|K(t)\| \leq C'\frac{\exp(-\beta' t)}{t^{\rho'}}, \quad C', \beta', \rho' = \mathrm{const}, \quad C' > 0, \quad \beta' > 0, \quad 0 \leq \rho' < 1 \tag{1.2}$$

In Eq. (1.1)

$$y = \int_0^t k(t-s)\varphi(x(s), s)ds \tag{1.3}$$

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and z is an analytic functional in the form of an absolutely convergent Frechet series

$$z(t) = \sum_{k=1}^{\infty} \sum_{j(k)=1}^{n} \int_0^t \dots \int_0^t K^{j(k)}(t-s_1, \dots, t-s_k) x_{j_1}(s_1) \dots x_{j_k}(s_k) ds_1 \dots ds_k$$
$$j(k) = j_1, \dots, j_k \tag{1.4}$$

where $k(t-s)$ is a continuous $n \times n$ matrix function, specified in the set

$$J_1' = \{(t, s) \in R^2 : 0 \le s < t < +\infty\}$$

while the continuous vector functions $K^{j(k)}(t-s_1, \dots, t-s_k)$ are specified in the set

$$J_k' = \{(t, s_1, \dots, s_k) \in R^{k+1} : 0 \le s_j < t < +\infty, j = 1, \dots, k\}$$

The vector function $\varphi(x, t)$ ($\varphi(0, t) \equiv 0$) of the class C^1 with respect to x in a certain neighbourhood $B'(x) = \{x \in R^n : \|x\| < \delta_1\}$ is also continuous and bounded with respect to t when $t \in R^+$.

We will assume that the integral kernels $k(t-s)$, $K^{j(k)}(t-s_1, \dots, t-s_k)$ satisfy the following inequalities

$$\|k(t-s)\| \le C \frac{\exp(-\beta(t-s))}{(t-s)^{\rho_0}} \tag{1.5}$$

$$\|K^{j(k)}(t_1, \dots, t_k)\| \le C \left(\frac{\exp(-\beta_1 t_1 - \dots - \beta_k t_k)}{(t_1 \dots t_k)^{\rho}} \right), \quad t_j = t - s_j \tag{1.6}$$

in which $C > 0$, $\beta > 0$, $\beta_j > 0$ ($j = 1, \dots, k$), $0 \le \rho_0 < 1$, $0 \le \rho < 1$ are constants, and a number β_0 exists such that $0 < \beta_0 \le \beta_j$ for all permissible j and k.

As regards the non-linear vector functions $\varphi(x, t) = \mathrm{col}(\varphi_1, \dots, \varphi_n)$, $F(x, y, z, t) = \mathrm{col}(F_1, \dots, F_n)$ in Eq. (1.1) and the representation (1.3), we will assume that the Lyapunov majorants [1]

$$\varphi^*(u) = \mathrm{col}(\varphi_1^*, \dots, \varphi_n^*), \quad F^*(u, v, w) = \mathrm{col}(F_1^*, \dots, F_n^*)$$

are constructed for them, and that these majorants satisfy the following inequalities for arbitrary ε such that $0 \le \varepsilon \le 1$:

$$\varphi_i^*(\varepsilon u) \le \varepsilon \varphi_i^*(u), \quad u \in B'(u), \quad i = 1, \dots, n$$
$$F_i^*(\varepsilon u, \varepsilon v, \varepsilon w) \le \varepsilon^{1+\delta} F_i(u, v, w), \quad \delta > 0, \quad (u, v, w) \in B(u, v, w) \tag{1.7}$$

The terms of Eq. (1.1) containing the parameter μ will be regarded as the perturbation; in this case, if non-linear terms are present in the perturbation we will assume that they relate to the function $F(x, y, z, t)$.

Note that in [2] it was proposed to use Volterra functionals of the form (1.4) in the mechanics of deformable bodies to describe the rheological properties of materials, thereby expressing the relation between the stress and the strain. Singularities of the form (1.2), (1.5) and (1.6) appeared, in particular, in integral kernels, characterizing the properties of such materials as, for example, polymers [3, 4].

Definitions. We will say that the continuous function $x(t)$, defined for $t \in R^+$, is exponentially limit periodic if it can be represented in the form

$$x(t) = x_p(t) + x_e(t) \tag{1.8}$$

where $x_p(t)$ is a periodic function with period $T > 0$, and the function $x_e(t)$ is such that $x_e(t) \to 0$ as $t \to +\infty$, where

$$\|x_e(t)\| \le C'' \exp(-\alpha' t), \quad C'' > 0, \quad \alpha' > 0 \tag{1.9}$$

We will denote the class of such functions by $\mathrm{lpe}(T, -\alpha')$.

We will also call the motion described by the function $x(t)$, limit periodic, if $x(t) \in \text{lpe}(T, -\alpha')$.

The class of functions $x_e(t)$, which satisfy inequality (1.9), will be denoted by $e_1(-\alpha')$.

In a similar way, if the continuous function $K(t_1, \ldots, t_k)$ $(t_j = t - s_j)$, specified in the set J'_k, satisfies inequality (1.6) when $0 < \beta_0 < \beta_j$ $(j = 1, \ldots, k)$, we will relate it to the class $e_k(-\beta_0)$.

Hence, the functions $\Phi(t)$ and $D_i(t)$ in Eq. (1.1), which occur in the persistent perturbation, are functions that are exponentially limit periodic and $\Phi(t), D_i(t) \in \text{lpe}(T, -\beta'')$ $(i = 1, 2, 3)$ for certain $\beta'' > 0$. Then, the periodic parts, $\Phi_p(t)$ and $D_{ip}(t)$, of the functions $\Phi(t)$ and $D_i(t)$ are bounded functions for $t \in R^+$, so that

$$\|\Phi_p(t)\|, \|D_{ip}(t)\| \le C_1, \quad C_1 = \text{const} > 0 \tag{1.10}$$

We will now refine the properties of the vectors $\varphi(x, t)$, $F(x, y, z, t)$ as functions of the variable t. We will assume that for all fixed $x \in B'(x)$ or $(x, y, z) \in B(x, y, z)$ the inclusion $\varphi(t) \in \text{lpe}(T, -\beta^0)$ holds or $F(x, y, z, t) \in \text{lpe}(T, -\beta^0)$ $(\beta^0 > 0)$ respectively, i.e. by equality (1.8) we have the representations

$$\varphi(x, t) = \varphi_p(x, t) + \varphi_e(x, t), \quad F(x, y, z, t) = F_p(x, y, z, t) + F_e(x, y, z, t) \tag{1.11}$$

We will investigate the structure of the general solution of Eq. (1.1), (1.3), (1.4) in the neighbourhood of the point $x = 0$ assuming that the unperturbed linear homogeneous equation with the lower limit of integration s, corresponding to (1.1), possesses the fundamental matrix $X(t-s)$ $(X(0) = E_n)$, subject to the inequality

$$\|X(t-s)\| \le C\exp(-\alpha(t-s)), \quad C, \alpha = \text{const} > 0 \tag{1.12}$$

i.e. the zeroth solution of the linearized homogeneous equation for Eq. (1.1) is asymptotically stable. We will consider the problem of the existence of limit periodic solutions of Eq. (1.1), (1.3), (1.4) with initial condition $x_0 = x(0) \in B''(x_0) \in B'(x_0)$. Note that, with the above assumptions (1.5)–(1.7) and (1.12), Eq. (1.1)–(1.4) is such that the point $x = 0$ is totally stable under persistent perturbations [5].

Theorem. Suppose the conditions of continuity or smoothness of the functions mentioned above are satisfied, and the functions occurring in the small perturbation satisfy the inclusion $\Phi(t), D_i(t) \in \text{lpe}(T, -\beta'')$ $(\beta'' > 0)$, and also for each fixed $(x, y, z) \in B(x, y, z)$, the property $\varphi(x, t), F(x, y, z, t) \in \text{lpe}(T, -\beta^0)$ $(\beta^0 > 0)$ holds. Suppose inequalities (1.2), (1.5), (1.6) and (1.12) holds, and Lyapunov majorants $\varphi^*(x)$, $F^*(x, y, z)$, which satisfy relations (1.7), exist.

Then $\delta > 0$ exists such that the general solution of Eq. (1.1), (1.3), (1.4) $x(t, x_0, \mu) \in \text{lpe}(T, -\gamma)$ for certain $\gamma > 0$ when $\|x_0\| < \delta$, $\mu < \delta$, i.e. this solution can be represented in the form

$$x(t, x_0, \mu) = x_p(t, \mu) + x_e(t, x_0, \mu) \tag{1.13}$$

where $x_p(t, \mu)$ is a periodic solution of the equation

$$\begin{aligned}
\frac{dx}{dt} &= Ax + \int_0^\infty K(s)x(t-s)ds + F_p(x, y, z, t) + \mu(\Phi_p(t) + D_{1p}(t)x + D_{2p}(t)y + D_{3p}(t)z) \\
y(t) &= \int_0^\infty k(s)\varphi_p(x(t-s), t-s)ds \\
z(t) &= \sum_{k=1}^\infty \sum_{j(k)=1}^n \int_0^\infty \ldots \int_0^\infty K^{j(k)}(s_1, \ldots, s_k)x_{j_1}(t-s_1)\ldots x_{j_k}(t-s_k)ds_1\ldots ds_k
\end{aligned} \tag{1.14}$$

and $F_p(x, y, z, t)$ and $\varphi_p(x(t), t)$ are the parts of the functions $F(x, y, z, t)$ and $\varphi(x(t), t)$ that are periodic in t in their representations (1.11).

Proof. We will construct the general solution of Eq. (1.1), (1.3), (1.4) in the neighbourhood of zero by the method of successive approximations, using for this purpose the integral equation

$$\begin{aligned}
x(t) &= X(t)x_0 + \int_0^t X(t-s)(F(x(s), y(s), z(s), s) + \\
&+ \mu(\Phi(s) + D_1(s)x(s) + D_2(s)y(s) + D_3(s)z(s))ds
\end{aligned} \tag{1.15}$$

which is equivalent to Eq. (1.1) together with the initial condition $x(0) = x_0$, and the integral representations (1.3) and (1.4).

Suppose $x^{(k)}(t), y^{(k)}(t), z^{(k)}(t)$ $(k = 1, 2, \ldots)$ are successive approximations, obtained from formulae (1.15), (1.3) and (1.4) by substituting relation (1.15), obtained in the previous step of the calculations of the quantities $x^{(k-1)}(t), y^{(k-1)}(t), z^{(k-1)}(t)$, into the right-hand side of relation (1.15), and the quantities $x^{(k)}(t)$ into the right-hand sides of relations (1.3) and (1.4), where we assume that

$$x^{(1)}(t) = X(t)x_0 + \mu\int_0^t X(t-s)\Phi(s)ds \tag{1.16}$$

We will denote the integral term in (1.16) by $X^{(1)}(t)$ and convert it, taking into account the structure of $\Phi(t)$, to the form

$$X^{(1)}(t) = \mu\int_0^t X(s)(\Phi_p(t-s) + \Phi_e(t-s))ds = \psi^{(1)}(t) + \chi^{(1)}(t)$$

where

$$\begin{aligned} \psi^{(1)}(t) &= \mu\int_0^\infty X(s)\Phi_p(t-s)ds \\ \chi^{(1)}(t) &= -\mu\int_t^\infty X(s)\Phi_p(t-s)ds + \mu\int_0^t X(s)\Phi_e(t-s)ds \end{aligned} \tag{1.17}$$

We will analyse the properties of functions (1.17). We have

$$\psi^{(1)}(t+T) = \mu\int_0^\infty X(s)\Phi_p(t+T-s)ds = \mu\int_0^\infty X(s)\Phi_p(t-s)ds = \psi^{(1)}(t) \tag{1.18}$$

i.e. $\psi^{(1)}(t)$ is a periodic function. For the vector function $\chi^{(1)}(t)$, taking into account the fact that $\Phi(t) \in \operatorname{lpe}(T, -\beta'')$ and, consequently, inequalities of the type (1.9), (1.10) and also (1.12) are satisfied, we obtain the limit

$$\begin{aligned} \|\chi^{(1)}(t)\| &\le \mu C\left(C_1\int_t^\infty \exp(-\alpha s)ds + C''\int_0^t \exp(-\alpha s)\exp(-\beta''(t-s))ds\right) = \\ &= \mu C\left[\frac{C_1}{\alpha}\exp(-\alpha t) + \frac{C''}{\beta''-\alpha}(\exp(-\alpha t) - \exp(-\beta'' t))\right] \end{aligned} \tag{1.19}$$

When obtaining inequality (1.19) we assumed $\alpha \neq \beta''$ for the sake of uniformity of estimation. This can always be achieved by changing one of the constants, for example, α, retaining the inequality of the form (1.12). We will put $0 < \gamma < \min(\alpha, \beta, \beta_0, \beta^0, \beta'')$; then, by inequality (1.19), $\chi^{(1)}(t) \in e_1(-\gamma)$ and, since $X(t) \in e_1(-\alpha)$, we obtain from relations (1.16), (1.18) and (1.19)

$$x^{(1)}(t) \in \operatorname{lpe}(T, -\gamma) \tag{1.20}$$

Consequently, if $x^{(1)}(t) = \operatorname{col}(x_1^{(1)}(t), \ldots, x_n^{(1)}(t))$, we can assume that

$$|x_i^{(1)}(t)| \le u_i^{(1)}(x_0, \mu) = \text{const}, \quad i = 1, \ldots, n$$

Consider the vector function $y^{(1)}(t)$, which, by relations (1.3), (1.11) and (1.20), can be written as follows:

$$y^{(1)}(t) = \int_0^t k(t-s)\varphi(x_p^{(1)}(s) + x_e^{(1)}(s), s)ds = Y_p^{(1)}(t) + Y_e^{(1)}(t)$$

where

$$Y_p^{(1)}(t) = \int_0^\infty k(s)\varphi_p(x_p^{(1)}(t-s), t-s)ds$$

$$Y_e^{(1)}(t) = -\int_t^\infty k(s)\varphi_p(x_p^{(1)}(t-s), t-s)ds + \tag{1.21}$$

$$+\int_0^t k(s)[\varphi(x_p^{(1)}(t-s) + x_e^{(1)}(t-s), t-s) - \varphi_p(x_p^{(1)}(t-s), t-s)]ds$$

The function $Y_p^{(1)}(t)$, specified by the convergent integral for $t \in R^+$, like the function $\psi^{(1)}(t)$ (1.17), is also periodic with period T.

We will show that $Y_e^{(1)}(t) \in e_1(-\gamma)$ for certain $\gamma' > 0$.

In fact, the function

$$\int_t^\infty k(s)\varphi_p(x^{(1)}(t-s), t-s)ds \tag{1.22}$$

which is continuous and bounded as $t \to +\infty$, decreases exponentially and approaches zero as $t \to +\infty$, and, in view of inequality (1.5), belongs to the class $e_1(-\beta)$. Using the Lipschitz condition for the function $\varphi(x, t)$ when $x \in B'(x)$, we estimate the integral

$$I(t) = \int_0^t k(s)[\varphi(x_p^{(1)}(t-s) + x_e^{(1)}(t-s), t-s) - \varphi_p(x_p^{(1)}(t-s), t-s)]ds$$

We have

$$\|I(t)\| \le \left\|\int_0^t k(s)[\varphi_p(x_p^{(1)}(t-s) + x_e^{(1)}(t-s), t-s) - \varphi_p(x_p^{(1)}(t-s), t-s)]ds\right\| +$$

$$+\left\|\int_0^t k(s)\varphi_e(x_p^{(1)}(t-s) + x_e^{(1)}(t-s), t-s)ds\right\| \le$$

$$\le C_L\int_0^t \|k(s)\|\left\|x_e^{(1)}(t-s)\right\|ds + C'\int_0^t \|k(s)\|\exp(-\gamma(t-s))ds \le \tag{1.23}$$

$$\le (C_LC_0 + C')C\exp(-\gamma t)\int_0^t \frac{\exp((\gamma-\beta)s)}{s^{\rho_0}}ds$$

where $C_L > 0$ is the Lipschitz constant, $C_0 > 0$ is the constant in an inequality of the type (1.9) for the function $x_e^{(1)}(t)$ and $C' > 0$ is an analogous constant for the function $\varphi_e(x, t)$ for all $x \in B'(x)$. Splitting the integral in the last of inequalities (1.23) into two parts with limits of integration 0 and 1, and also 1 and t, we obtain an estimate of the type (1.9). Consequently $Y_e^{(1)}(t) \in e_1(-\gamma)$ and $y^{(1)}(t) \in \mathrm{lpe}(T, -\gamma)$.

We will analyse the structure of the vector function $z^{(1)}(t)$, which is the series (1.4), in which the functions $x_j(s)$ take the values $x_j^{(1)}(s)$. We will denote the term of this series with the superscript $j(k)$ by $I^{j(k)}(t)$ and show that $I^{j(k)}(t) \in \mathrm{lpe}(T, -\gamma)$, i.e. $I^{j(k)}(t) = I_p^{j(k)}(t) + I_e^{j(k)}(t)$. We will represent the function

$$I^{j(k)}(t) = \int_0^t\ldots\int_0^t K^{j(k)}(t-s_1, \ldots, t-s_k)(x_{j_1p}^{(1)}(s_1) + x_{j_1e}^{(1)}(s_1))\ldots(x_{j_kp}^{(1)}(s_k) + x_{j_ke}^{(1)}(s_k))ds_1\ldots ds_k$$

in the form of the sum of the functions

$$I_p^{j(k)}(t) = \int_0^t \ldots \int_0^t K^{j(k)}(t-s_1, \ldots, t-s_k) x_{j_1 p}^{(1)}(t-s_1) \ldots x_{j_k p}^{(1)}(t-s_k) ds_1 \ldots ds_k$$

$$I_e^{j(k)}(t) = I^{j(k)}(t) - I_p^{j(k)}(t)$$

The periodicity of the first of these is obvious, while the property $I_e^{j(k)}(t) \in e_1(-\gamma)$ establishes the same property of the function $Y_e^{(1)}(t)$ (1.21), since when estimating the function $I_e^{j(k)}(t)$, the multiple integrals occurring in it transform, in view of inequality (1.6), in the product of multiple integrals of the form (1.22) and differences of the integrals, which occur in the upper row of inequality (1.23). Hence, the proof of this property is carried out in a similar way. Thus, the series $z^{(1)}(t) \in \text{lpe}(T, -\gamma)$.

In the second approximation, the functions $x^{(2)}(t)$, $y^{(2)}(t)$ and $z^{(2)}(t)$ are given by the relation

$$x^{(2)}(t) = X(t)x_0 + \int_0^t X(t-s)[\mu(\Phi(s) + D_1(s)x^{(1)}(s) + D_2(s)y^{(1)}(s) + D_3(s)z^{(1)}(s)) + F(x^{(1)}(s), y^{(1)}(s), z^{(1)}(s), s)]ds \tag{1.24}$$

which follows from formula (1.15), and relations (1.3) and (1.4), on the right-hand sides of which we put $x(t) = x^{(2)}(t)$. Since the function $F(x, y, z, t)$ is exponentially limit periodic in t for fixed x, y, z and $x^{(1)}(t), y^{(1)}(t), z^{(1)}(t) \in \text{lpe}(T, -\gamma)$, the function $F(x^{(1)}(t), y^{(1)}(t), z^{(1)}(t), t)$ will be exponentially limit periodic, as also the functions $\Phi(t)$ and $D_i(t)$. Hence, repeating the previous discussions, it can be shown that $x^{(2)}(t), y^{(2)}(t), z^{(2)}(t) \in \text{lpe}(T, -\gamma)$. By virtue of the properties of the function F, the integral operator on the right-hand side of relation (1.24) will be compressive.

In a similar way, in the general case, one can establish the property of the functions $x^{(k)}(t)$, $y^{(k)}(t)$, $z^{(k)}(t) \in \text{lpe}(T, -\gamma)$, if $x^{(k-1)}(t), y^{(k-1)}(t), z^{(k-1)}(t) \in \text{lpe}(T, -\gamma)$ for certain $\gamma > 0$. It was shown in [5, Theorem 2] that the successive approximations $x^{(k)}(t), y^{(k)}(t)$ and $z^{(k)}(t)$ converge when $\|x_0\| < \delta$, $\mu < \delta$ for certain $\delta > 0$ to the functions $x(t)$, $y(t)$ and $z(t)$ respectively, which are the solution of Eq. (1.1), (1.3), (1.4). This is established by constructing the majorizing equation for

$$u(x_0, \mu) \gg x(t, x_0, \mu), \quad \upsilon(x_0, \mu) \gg y(t, x_0, \mu), \quad w(x_0, \mu) \gg z(t, x_0, \mu)$$

where

$$u(x_0, \mu) = \lim_{k \to +\infty} u^{(k)}(x_0, \mu), \quad \upsilon(x_0, \mu) = \lim_{k \to +\infty} \upsilon^{(k)}(x_0, \mu), \quad w(x_0, \mu) = \lim_{k \to +\infty} w^{(k)}(x_0, \mu)$$

and $u^{(k)}$, $\upsilon^{(k)}$ and $w^{(k)}$ are majorizing sequences for $x^{(k)}(t, x_0, \mu)$, $y^{(k)}(t, x_0, \mu)$ and $z^{(k)}(t, x_0, \mu)$. The last functions belong to the class $\text{lpe}(T, -\gamma)$ and, as follows from the construction of the successive approximations $x_p^{(k)}$, $y_p^{(k)}$ and $z_p^{(k)}$, the periodic functions $x_p(t, \mu)$, $y_p(t, \mu)$ and $z_p(t, \mu)$ are a solution of Eq. (1.14) when $x_0 = 0$ and $\mu < \delta$, while the solution itself $x(t, x_0, \mu)$ is represented in the form (1.13).

It should be noted that the periodic solutions of Volterra type integro-differential equations with infinite after-effect, to which Eq. (1.14) belongs, have been considered in many publications (see, for example, [4–9]). In [7], where general problems of the theory of periodic solutions were considered and particular equations relating to certain applications were investigated, there is a considerable bibliography on this problem. An investigation has been carried out on periodic solutions for integro-differential equations with an upper limit of integration in the form of periodic functions in [8]. It was suggested in [6, 7] that periodic solutions can be represented by Fourier series. The proof of the convergence of the successive approximations was based on the method of majorizing equations in [6].

2. LIMIT PERIODIC MOTIONS OF A WING

We will consider the problem of the rotational motion of a wing (a thin rigid plate) around a longitudinal horizontal axis when there is an unsteady air flow about it [10]. We will carry out the investigation using the model of unsteady flow proposed by Belotserkovskii [11], based on the introduction of integral terms into the expressions for the aerodynamic forces and their moments acting on the wing.

We will denote the angle of rotation of the plate by ϑ, measured from the horizontal fixed axis Ox_1 in the vertical plane (see Figure 1). Figure 1 shows a section of the wing by a vertical plane passing

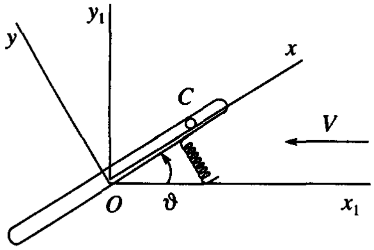

Fig. 1

through the centre of mass C of the wing. The point C has coordinates x_0, y_0 in a system of coordinates Oxy, permanently connected with the plate. The unperturbed flow is directed horizontally and has constant velocity $\mathbf{V}_0$ parallel to the Ox_1 axis. Viscoelastic forces, which, it can be assumed, are produced by a viscoelastic spring, act on the mounting of the wing. The moment of these forces is perpendicular to the vertical plane Ox_1y_1 and has a value L. We will represent the relation between L and the angle ϑ, characterizing the deformation of the spring, by a functional in the form of a Volterra–Frechet series [2, 3] of the type (1.4), assuming that the moment only changes sign when the sign of the deformation $\vartheta' = \vartheta_0 + \vartheta$ changes

$$L = -l\vartheta' + \int_0^t L'(t-s)\vartheta'(s)ds + \int_0^t\int_0^t\int_0^t L^{(3)}(t-s_1, t-s_2, t-s_3)\vartheta'(s_1)\vartheta'(s_2)\vartheta'(s_3)ds_1ds_2ds_3 + \dots \tag{2.1}$$

where l is the modulus of elasticity for torsion, ϑ' is the total deformation of the spring and the constant ϑ_0 is chosen so that the value $\vartheta = 0$ is the equilibrium position of the plate under steady flow when the spring has no rheological properties. The kernels of the relaxation $L'(t)$ and $L^{(k)}$ $(t_1, \dots, t_k)$ $(k = 3, 5, \dots)$ in representation (2.1) are continuous functions which satisfy inequality (1.6).

We will assume that the angle of attack (the angle between the plane of the wing and the vector of the relative velocity of the flow at the point A of the leading edge) is expressed in terms of the angle ϑ. Then the moment M of the aerodynamic forces acting on the wing in the case of steady flow [11, 10] can be written by separating out in explicit form the non-linear terms up to the third order inclusive,

$$M = m_0 + m_1\dot{\vartheta} + m_2\vartheta + m'\vartheta^3 + m''\dot{\vartheta}^3 + \int_0^t I_1(t-s)\ddot{\vartheta}(s)ds + \int_0^t I_2(t-s)\dot{\vartheta}(s)ds + I_1(t)\dot{\vartheta}(0) + I_2(t)\vartheta(0) + M' \tag{2.2}$$

where m_0, m_1, m_2, m' and m'' are constants, and the functions $I_1(t), I_2(t) \in C^1$ and M' are non-linear terms of higher than the third order.

The velocity $\mathbf{V}$ of the unperturbed flow will be assumed to be directed along the vector $\mathbf{V}_0$ and we will also assume that the algebraic values V and V_0 of these vectors are connected by the relation

$$V = V_0 + \mu\upsilon(t) \tag{2.3}$$

where $\mu \ll 1$ and $\upsilon(t)$ is a continuous function of the class lpe$(T, -\gamma)$ $(\gamma > 0)$.

For the perturbed flow the moment $\tilde{M}$ of the aerodynamic forces, which depend on V (2.3), will be specified by formula (2.2) in which the constants m_j $(j = 0, 1, 2)$, m' and m'' and the functions $I_i(t)$ $(i = 1, 2)$ are replaced, respectively, by the quantities

$$\tilde{m}_j(t) = (1+\mu\psi_j(t))m_j, \quad \tilde{m}'(t) = (1+\mu\psi'(t))m'$$
$$\tilde{m}''(t) = (1+\mu\psi''(t))m'', \quad \tilde{I}_i(t) = (1+\mu\chi_i(t))I_i(t) \tag{2.4}$$

where the functions $\psi_j(t)$, $\psi'(t)$, $\psi''(t)$, $\chi_i(t) \in$ lpe(T, γ).

We will write the equation of the rotational motions of the plate in the form of the system ($x_1 = \dot{\vartheta}$, $x_2 = \vartheta$)

$$\frac{dx_1}{dt} = \sum_{j=1,2}\left[a_j x_j + \int_0^t K_j(t-s)x_j(s)ds + \mu\left(\int_0^t K'_j(t,s)x_j(s)ds + a'_j(t)x_j\right)\right] + \mu\Phi(t) + F$$
$$\frac{dx_2}{dt} = x_1 \tag{2.5}$$

in which, based on formulae (2.1), (2.2) and (2.4)

$$a_1 = (m_1 + I_1(0))/I, \quad a_2 = (mgx_0\sin\vartheta_0 + mgy_0\cos\vartheta_0 - l + m_2 + I_2(0))/I$$
$$a'_1(t) = a_1\psi_1(t), \quad a'_2(t) = (m_2 + I_2(0))\psi_2(t)/I$$
$$K_1(t) = (dI_1(t)/dt)/I, \quad K_2(t) = (dI_2(t)/dt + L'(t))/I$$
$$K'_1(t,s) = K_1(t-s)\chi_1(t), \quad K'_2(t,s) = K_2(t-s)\chi_2(t) + L''(t,s)$$
$$\Phi(t) = (m_0\psi_0(t) + \varphi'(t))/I, \quad F = F_2 + F_3 + F' \tag{2.6}$$
$$F_2 = \frac{1}{I}\left[\frac{1}{2}mg(x_0\cos\vartheta_0 - y_0\sin\vartheta_0)x_2^2 + \int_0^t\int_0^t K^{(2)}(t,s_1,s_2)x_2(s_1)x_2(s_2)ds_1ds_2\right]$$
$$F_3 = \frac{1}{I}\left[\tilde{m}''x_1^3 + \tilde{m}'x_2^3 - \frac{1}{6}mg(x_0\sin\vartheta_0 + y_0\cos\vartheta_0)x_2^3 + \right.$$
$$\left. + \int_0^t\int_0^t\int_0^t K^{(3)}(t,s_1,s_2,s_3)x_2(s_1)x_2(s_2)x_2(s_3)ds_1ds_2ds_3\right]$$

where mg is the weight of the body, I is its moment of inertia about the axis of rotation passing through the point O, and F' is the set of terms of higher than the third order. The functions $\varphi'(t)$, $L''(t, s)$, $K^{(2)}(t, s_1, s_2)$ and $K^{(3)}(t, s_1, s_2, s_3)$, which are not given in detail here, are found from formula (2.1).

It is easy to construct expressions for these, for example, in the special case [3, p. 607] when the integral kernels in (2.1) have the following structure

$$L^{(2k+1)}(s_1, s_2, \ldots, s_{2k+1}) = l_{2k+1}\prod_{i=1}^{2k+1}\tilde{L}(s_i), \quad l_{2k+1} = \text{const}, \quad k = 1, 2, \ldots$$

and relation (2.1) then takes the form

$$L = -l\vartheta' + y' + \sum_{i=1}^{\infty} l_{2i+1}\tilde{y}^{2i+1} \equiv -l\vartheta' + y' + \tilde{y}^3 S(\tilde{y})$$
$$y' = \int_0^t L'(t-s)\vartheta'(s)ds, \quad \tilde{y} = \int_0^t \tilde{L}(t-s)\vartheta'(s)ds$$

where $S(\tilde{y})$ is a holomorphic function. In particular, if $\vartheta_0 = 0$, we have

$$\varphi'(t) \equiv 0, \quad L''(t,s) \equiv 0, \quad K^{(2)}(t,s_1,s_2) \equiv 0$$

and the integral term in the expression for F_3 (2.6) is reduced to the form $l_3\tilde{y}^3/I$.

We will assume that the integral kernels $K_j(t)$ ($j = 1, 2$) (2.6) satisfy inequality (1.2). Terms of order μ, linear in x_j, will be assumed to be the perturbation in Eqs (2.5). The function $\varphi'(t)$ can be represented in the form $\varphi'(t) = c' + \varphi'_e(t)$, where $c' = \text{const}$ and $\varphi'_e(t) \in e(-\gamma')$ for certain $\gamma' > 0$, the function $\Phi(t) \in \text{lpe}(T, -\gamma)$ and all the integral kernels $K^{(k)}(t, s_1, \ldots, s_k) \in e'_k(-\beta_0)$.

For the unperturbed equation (2.5) we will set up the characteristic equation

$$d(\lambda) \equiv \lambda^2 - a_1\lambda - a_2 - \lambda K_1^*(\lambda) - K_2^*(\lambda) = 0 \tag{2.7}$$

where $K_i^*(\lambda)$ is the Laplace transform for the function $K_i(t)$ $(i = 1, 2)$.

Suppose the characteristic equation (2.7) has a finite number of roots λ_j $(j = 1, \dots, L)$ in the complex half-plane $\mathrm{Re}\lambda > -\beta'$ and $\mathrm{Re}\lambda_j < 0$. Then condition (1.12) is satisfied. By virtue of the assumptions made in this section the theorem of Section 1 holds for Eqs (2.5) and (2.6) and, consequently, under the effect of the periodic part of the perturbation of the flow in the limit periodic vibrations of the wing will become established. These periodic vibrations, by Eq. (1.14), correspond to the periodic solution of the equation

$$\begin{aligned}\ddot{\vartheta} = {} & a_1(1 + \mu\psi_{1p}(t))\dot{\vartheta} + (a_2 + \mu a'_{2p}(t))\vartheta + \int_0^\infty [(K_1(s) + \mu\tilde{K}_1(t, s))\dot{\vartheta}(t-s) + \\ & + (K_2(t, s) + \mu\tilde{K}_2(t, s))\vartheta(t-s)]ds + \mu\Phi_p(t) + F\end{aligned} \tag{2.8}$$

where, in all the integrals occurring in F, the upper limits of integration of t are replaced by ∞. In Eq. (2.8) and, consequently, in formulae (2.6) the functions $\psi(t)$, $\chi(t)$ with different subscripts, and also $\Phi_p(t)$, are replaced by $\psi_p(t)$, $\chi_p(t)$ respectively with same subscripts and $\Phi_p(t)$, i.e. by the periodic parts of these functions. Moreover, in Eq. (2.8) $\tilde{K}_i(t, s)$ are functions into which the integral kernels $K'_i(t, s)$ $(i = 1, 2)$ transform.

In the first approximation, the periodic solution of Eq. (2.8) has the form

$$\vartheta^{(1)}(t) = \mu\int_0^\infty x_{21}(s)\Phi_p(t-s)ds \tag{2.9}$$

where $x_{21}(t)$ is an element of the fundamental matrix $X(t) = (x_{ij}(t))$ $(i, j = 1, 2)$ in inequality (1.12).

Solution (2.9) can be sought in another form [6, 7], if the function $\Phi_p(t)$ can be represented by an absolutely convergent Fourier series, so that

$$\Phi_p(t) = \sum_{k=0}^{\infty}(b_k\sin(k\omega t) + c_k\cos(k\omega t)), \quad \omega = \frac{2\pi}{T} \tag{2.10}$$

Then, specifying the required function $\vartheta^{(1)}(t)$ by a Fourier series of the form (2.10) with coefficients b'_k and c'_k, to determine these constants for each k we will have a system of linear algebraic equations, the determinant of which is equivalent to the quantity $d(ik\omega)$ in relation (2.7) and is non-zero for all k by virtue of the assumption made that there are no pure imaginary roots in the characteristic equation. Consequently, this series can be constructed and will be absolutely convergent together with the Fourier series for the first and second derivatives, provided the function $\Phi(t)$ possesses, for example, a piecewise-continuous first derivative.

Using formulae (2.6) for the non-linear terms, we can calculate the third approximation of the periodic mode, which satisfies Eq. (2.8) (up to terms of the order of μ^3 inclusive).

The rate at which the limit periodic solutions tend to periodic solutions is determined by the real parts of the roots of the characteristic equation and the exponents in the integral kernels, and also in the limit periodic functions, by which the perturbation is specified.

We will consider the example of the analytical determination of an estimate for these real parts and thereby of the quantity α in inequality (1.12). Suppose

$$l_i(t) = d_{i1}\exp(-\gamma_1 t) + d_{i2}\exp(-\gamma_2 t), \quad d_{ij} = \mathrm{const}, \quad \gamma_i = \mathrm{const} > 0, \quad i, j = 1, 2 \tag{2.11}$$

Taking representation (2.11) into account, we write characteristic equation (2.7) in the form

$$\lambda^2 - a_1\lambda - a_2 + \sum_{i,j=1}^{2}\frac{d_{ij}\gamma_j\lambda^{2-i}}{\gamma_j + \lambda} = 0 \tag{2.12}$$

We will estimate the roots of Eq. (2.12) assuming that

$$d_{ij} = \mu\tilde{d}_{ij}, \quad 0<\mu \ll 1, \quad i,j = 1,2 \tag{2.13}$$

and the values of γ_i are fairly high and that the Routh–Hurwitz conditions are satisfied, i.e. all the roots λ_k (k = 1, 2, 3, 4) (numbered in order of increase of the real part) lie in the left half-plane. We will represent λ_k in the form

$$\lambda_k = \sum_{p=0}^{\infty} \lambda_k^{(p)}\mu^p \tag{2.14}$$

We have

$$\lambda_{1,2}^{(0)} = \frac{1}{2}(a_1 \pm \sqrt{a_1^2+4a_2}), \quad \mathrm{Re}\lambda_{1,2}^{(0)}<0, \quad a_1<0, \quad a_2<0$$

We will consider the case of the complex-conjugate quantities $\lambda_1^{(0)}$, $\lambda_2^{(0)}$ and we will obtain the constant u_0 for the estimate $|\mathrm{Re}\lambda_1| > u_0$. To do this we will obtain the upper limit for the moduli of the roots λ_3 and λ_4 of Eq. (2.12). We will denote by $\upsilon_j(\mu)$ ($\upsilon_j(0) = 0, j = 3, 4$) the series, which majorize the power series in μ for the quantities $\lambda_j - \lambda_j^{(0)}$, in which we put, corresponding to Eq. (2.12), reduced to polynomial form, $\lambda_j^{(0)} = -\gamma_{j-2}$ ($\gamma_2 > \gamma_1$). To obtain $\upsilon_3(\mu)$ we introduce the majorizing equation

$$b_1 = |2\gamma_1 - a_1|\upsilon_3 + \upsilon_3^2 + \mu\sum_{i=1}^{2}(\gamma_1+\upsilon_3)^{2-i}\left(\frac{|\tilde{d}_{i1}|\gamma_1}{\upsilon_3} + \frac{|\tilde{d}_{i2}|\gamma_2}{b_0-\upsilon_3}\right)$$
$$b_0 = \gamma_2-\gamma_1, \quad b_1 = |\gamma_1^2+a_1\gamma_1-a_1| \neq 0 \tag{2.15}$$

To obtain an estimate in explicit form we will reinforce the majorizing equation (2.15), using the following majorants for terms that are independent of μ, instead of υ_3 and υ_3^2

$$\upsilon_3 \ll \frac{b_0\upsilon_3}{b_0-\upsilon_3}, \quad \upsilon_3^2 \ll \frac{b_0^2\upsilon_3}{b_0-\upsilon_3}$$

We will denote by υ_{3*} the least positive root of the new (quadratic) equation. Then

$$|\lambda_3| < \lambda_1 + \upsilon_{3*} \tag{2.16}$$

Proceeding in a similar way, we obtain the corresponding root $\upsilon_{4*} > 0$ and estimate of the real root of Eq. (2.12)

$$|\lambda_4| < \gamma_2 + \upsilon_{4*} \tag{2.17}$$

On the basis of Eq. (2.12), reduced to polynomial form, we have

$$\left|\sum_{s=1}^{4}\lambda_s\right| = -a_1+\gamma_1+\gamma_2 \tag{2.18}$$

Since $\lambda_1 + \lambda_2 = 2\mathrm{Re}\lambda_1$, using relations (2.16)–(2.18), we obtain the required estimate

$$|\mathrm{Re}\lambda_1| > u_0 = \frac{1}{2}(-a_1-\upsilon_{3*}-\upsilon_{4*}) \tag{2.19}$$

In a similar way we can construct an estimate for $|\mathrm{Re}\lambda_1|$ in the case when the integral kernels in representation (2.2) contain an arbitrary finite number of exponential functions and are specified by the formula

$$I_i(t) = \mu\sum_{s=1}^{p}\tilde{d}_{is}\exp(-\gamma_i t), \quad \tilde{d}_{is} = \mathrm{const}, \quad \gamma_i = \mathrm{const}$$

An estimate of the type (2.19) enables us to avoid the numerical determination of the roots of the characteristic equation.

In the more general problem of the three-dimensional motions of a wing acted upon by an unsteady free airstream [10], the wing is represented by a rigid body (a thin plate) with one fixed point. The mounting of the wing is modelled by a viscoelastic spring, and the stress-strain relationship is specified in the general case by the Volterra–Frechet series (2.1). The system has three degrees of freedom and its motion is described by Volterra type integro-differential equations (1.1). The moments of the aerodynamic forces, taking into account the perturbation of the flow velocity, are given by formulae of the type (2.2) and (2.4). If the perturbation of the flow is described by exponentially limit periodic functions of time, then, according to the theorem proved above, limit periodic motions of the wing are established, which, as time passes, approach more and more to periodic.

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